Solutions to Selected Problems from West, Installment #1

1.2.16 Suppose edge e appears an odd number of times in a closed walk W. Prove that W contains a cycle using e.

Proof: If W is a cycle, then we're done, so assume not. Since e appears an odd number of times in W, it follows that e is not a cut-edge of W. Let x and y be the endpoints of e. Since e is not a bridge, there exists an x, y-path P in W that does not use e, but then $P + e$ is a cycle containing e. □

1.2.17 Let G be the graph whose vertex set if the set of permutations of $\{1, \ldots, n\}$, with two permutations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ adjacent if they differ by interchanging a pair of adjacent entries. Prove that G is connected.

Proof: Let $x = a_1, \ldots, a_n$ and $y = b_1, \ldots, b_n$ be vertices in G. Since x and y are permutations of $\{1, \ldots, n\}$, it follows that $y = x\phi$, where $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a bijection. By a standard result from the lore of permutations, ϕ can be factored as a product of transpositions (pair interchanges); if k is the number of transpositions in the factorization of ϕ, then there is a path of length k connecting A to B. Thus G is connected. □

1.2.24 Prove that a finite graph having at least one edge contains at least two vertices that are not cut-vertices.

Proof: Let G be a finite graph with at least one edge. We may assume that G is connected. Let P be a longest path in G, with endpoints x and y. By way of contradiction, suppose x is a cut-vertex. Then there exist vertices s, t in G, adjacent to x, with the property that every s, t-path uses x. Clearly at most one of s, t lies on P. Suppose that t does not lie on P. Then $P + xt$ is a path in G longer than P, but this is impossible by our choice of P. Thus x is not a cut-vertex. By interchanging labels x and y, we see that y is also not a cut-vertex. □

2.3.21 Develop an efficient algorithm that, given a graph as input, decides whether the graph is bipartite. The graph is given by adjacency matrix or adjacency lists. The algorithm should not need to look at any edge twice.

Solution: (a sketch) An enhanced BFS will do the trick. Without loss of generality, assume that the input graph is connected. We keep track of the distance from the starting vertex to each other vertex. For example, if we begin at u we "mark" u by setting $d(u) = 0$, and place u on a queue. When we remove a vertex v from the queue, we first check to see that $d(v)$ has parity opposite that of $d(x)$ for each marked vertex $x \in N(v)$. If this fails, the graph is not bipartite and we quit. Otherwise, for each unmarked $x \in N(v)$, we mark x by setting $d(x) = 1 + d(v)$, and place x in the queue. If we ever go to the queue and find it empty, the parity test has never failed and the graph is bipartite.

2.3.13 Suppose T is a MST in G, and that T' is another spanning tree in G. Prove that T' can be transformed into T by a sequence of steps that exchange one edge of T' for one edge of T such that the edge set is always a spanning tree and such that the total weight never increases.

Proof: Let T and T' be as described. If $T = T'$, we're done: the sequence in question has length zero. Suppose $T \neq T'$, and that the result holds for all trees T'' that have more edges than T' in common with T. Let e be an edge of T that is not an edge of T'. Then $T + e$ contains a cycle C. Among all edges of C that are not edges in T (there must be at least one), choose e' with maximum weight. It must be that $w(e') > w(e)$, since otherwise $T - e + e'$ is a spanning tree with lighter weight than T. But then $T'' = T' + e - e'$ is a spanning tree with lighter weight than T'. The result follows inductively. $\square$

2.4.1 Prove, or provide a counterexample: There is no connected Eulerian simple graph with an even number of vertices and an odd number of edges.

Counterexample: Construct a copy of C_6, with vertices consecutively labeled as $v_1, v_2, \ldots, v_6$. Add edges v_1v_3, v_3v_5, and v_5v_1. A general procedure to construct a counterexample of arbitrary even order $n > 4$ is easy to devise: start with C_n, and insert a collection of additional edges that induce an odd cycle.

2.4.12 Prove that every undirected graph G has an orientation D such that $|d^+(v) - d^-(v)| \leq 1$ for every vertex $v \in V(G)$.

Proof: Let G be an undirected graph. If G is Eulerian, orient the edges of G as indicated by an Eulerian tour of G. In this case, $d^+(v) = d^-(v)$ for every vertex $v \in V(G)$. Suppose, then, that G is not Eulerian. It follows that G contains some even number of odd vertices. Insert temporary edges joining pairs of odd vertices. The resulting graph, call it G', is Eulerian. Orient the edges of G' as in the previous case. The temporary edges have become temporary arcs. Delete them. The result follows. $\square$

3.1.13 Let M and M' be matchings in an X, Y-bigraph G. Suppose that M saturates $S \subseteq X$ and M' saturates $T \subseteq Y$. Prove that G has a matching that saturates $S \bigcup T$.

Proof: We construct a matching M'' that saturates $S \bigcup T$, using selected edges of $M \bigcup M'$. Initially set $M'' := \emptyset$. Consider the subgraph H induced by $M \bigcup M'$. A component in H is either an isolated edge, an even cycle in which edges alternate between M and M' (an "alternating cycle"), or a path in which edges alternate between M and M' (an "alternating path"). For each isolated edge in $e \in M \bigcup M'$, set

$M'' := M'' + e$. For each alternating cycle C in H, set $M'' := M'' + (E(C) \bigcap M)$. For each alternating path P of odd length, set $M'' := M'' + (E(P) \bigcap M)$ if the endpoints of P are M-saturated; otherwise set $M'' := M'' + (E(P) \bigcap M')$. For each alternating path P of even length, set $M'' := M'' + (E(P) \bigcap M)$ if both ends of P lie in X, and set $M'' := M'' + (E(P) \bigcap M')$ if both ends of P lie in Y. Since we did not take both M- and M'- edges from any component of H, M'' is a matching in G. It remains to show that $S \bigcup T$ is saturated. Let $v \in S \bigcup T$. If $v \in S \bigcup T$ is saturated by an isolated edge $e \in E(H)$, then since $e \in M''$ we know that v is M''-saturated. If v lies on an alternating cycle C in H, then v is saturated by one edge from each matching; the choice of $E(C) \bigcap M$ is arbitrary, and v is saturated by M''. If v is an internal vertex on an alternating path P of odd length, then v is saturated by both M and M', so we must consider only the case in which v is an endpoint of P. Since P has odd length, v and the opposite endpoint of P are either both $(M - M')$-saturated or both $(M' - M)$-saturated, and we choose accordingly to ensure that the endpoints of P are M''-saturated. Finally, suppose that v lies on an alternating path P of even length. As in the odd case, if v is an internal vertex on P then v is clearly M''-saturated, so suppose that v is an endpoint of P. If $v \in S$, then P is of the form $v = x_1, y_1, x_2, \ldots, y_{k-1}, x_k$ for some $k \geq 2$. It follows that $x_1 y_1 \in M$, and that $x_k \in X - S$; since the M-edges of P saturate every vertex of P except x_k, these are the edges that we place in M''. Similarly, if $v \in T$, then P is of the form $v = y_1, x_1, y_2, \ldots, x_{k-1}, y_k$ for some $k \geq 2$, $y_1 x_1 \in M'$, and $y_k \in Y - T$. Taking the M' edges of P saturates every vertex of P except y_k, and we're done. $\square$

3.1.25 Let Q be an $n \times n$ doubly stochastic matrix. Show that Q can be expressed as a convex combination of permutation matrices.

Before proving the claim, we establish the following lemma:

Lemma 1 *Let Q be an $n \times n$ doubly stochastic matrix. Then there exists an $n \times n$ permutation matrix P with the property that, for all i, j, $p_{ij} \neq 0 \Rightarrow q_{ij} \neq 0$.*

Proof: Construct a graph $G = (R, C, E)$, where $r_i c_j \in E$ iff $q_{ij} \neq 0$. Let M be a matching in G. Set $p_{ij} = \begin{cases} 1; r_i c_j \in M \\ 0; \text{otherwise.} \end{cases}$ It is not hard to see that $P = (p_{ij})$ is a permutation matrix if and only if M is a perfect matching. It suffices to show that such a matching exists. We do so by showing that G satisfies Hall's condition. So let $S \subseteq R$. Suppose that $|N(S)| < |S|$. Then (relabeling the vertices if necessary) there exist $t < s$ such that $S = \{r_1, \ldots, r_s\}$ and $N(S) = \{c_1, \ldots, c_t\}$. Since the only nonzeros in rows $1, 2, \ldots, s$ of Q are in columns $1, 2, \ldots, t$, and since Q is doubly stochastic, it follows that $\sum_{i=1}^{s} \sum_{j=1}^{t} q_{ij} = \sum_{i=1}^{s} 1 = s$. But then the average of column sums $1, 2, \ldots, t$ is at least

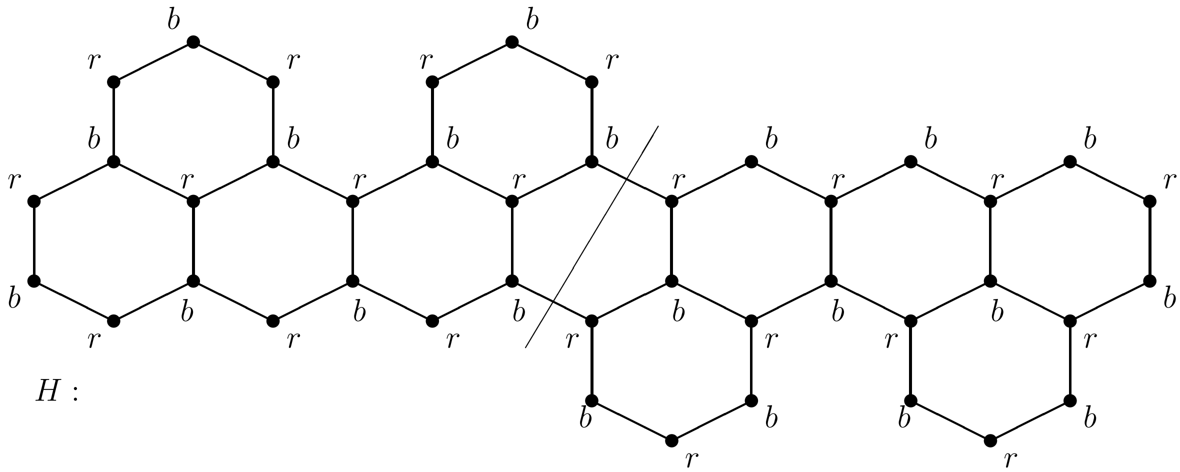

Figure 1: Figure for problem 3.1.28

$r/t > 1$, a contradiction. So $|N(S)| \geq |S|$ for each $S \subset R$. A similar argument applies to subsets $S \subseteq C$. So at least one perfect matching in G exists, and the result follows. $\square$

We now prove (3.1.15). Let Q be an $n \times n$ doubly stochastic matrix. If Q contains precisely n nonzeros, then Q is itself a permutation matrix. Suppose, then, that Q contains k nonzeros ($n < k \leq n^2$), and assume that the result holds for all nonnegative matrices with fewer than k nonzeros and with constant row and column sums. By Lemma 1, there exists a permutation matrix P whose nonzeros correspond to nonzeros in Q. Let ϵ be the least nonzero entry in Q corresponding to a nonzero in P. By the induction hypothesis, there exist permutation matrices $P_1, P_2, \ldots, P_m$ and positive coefficients $c_1, c_2, \ldots, c_m$ such that $Q - \epsilon P = \sum_{i=1}^{m} c_i P_i$, and it follows that

$Q = \sum_{i=1}^{m} c_i P_i + \epsilon P$, a convex combination of permutation matrices. $\square$

3.1.28 We are to find either a perfect matching in the graph H, below, or a simple proof that none exists. The reality is that none exists. The graph is bipartite, and can be labeled as shown using colors red and blue. There are 21 vertices of each color, so at first glance it appears that a perfect matching might exist. Such a matching would necessarily contain 21 edges. But consider the edge cut shown by the line running southwest to northeast and bisecting the graph. The ten blue vertices to the left of the cut and the ten red vertices to the right constitute a vertex covering of cardinality twenty, so by the König-Egerváry theorem no perfect matching can exist. $\square$